

Rotating relativistic superfluid

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Abstract

Relativistic equation of state and velocity comparable with the speed of light are included in consideration of a superfluid rotating in a cylindrical container. Minimizing the free energy, we derive the equation of motion. It admits an analytic solution, the solid-body rotation inside and irrotational motion near the walls of the vessel, providing the vortex quantum is not extremely high that is satisfied for real astrophysical objects. The relativistic velocity of the vessel and the relativistic equation of state results to the deviation of the angular velocity of the solid-body motion inside and that of the vessel. The boundary between the solid-body and irrotational motion is also shifted sufficiently leading to a difference between the total angular momentum of superfluid and the normal matter.

1 Introduction

The purpose of the present study is to investigate the dynamics of the relativistic rotating superfluid. It will be a generalization of the non-relativistic theory [1]. The medium is usually called relativistic in two senses: if it has a relativistic equation of state or when it flows at a relativistic velocity. Both

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conditions (especially the first one) take place in neutron stars whose massive interior is composed of a superfluid nuclear matter [2]. A superfluid with quantum vortices has a third measure of relativity; it corresponds to the vortices - whether they are relativistic [3]. This possibility pertains better to the string fluids rather than to the real neutron star matter, although some small relativistic effects are also feasible here. We do not omit them immediately from the beginning and do not discuss more than it is necessary in the present paper with the aim to find possible applications to the neutron stars.

The signature of the Minkowsky space with a metric tensor $diag(\{-+++\})$ and (if not specified otherwise) the natural system of units ($\hbar = c = 1$) are used in the paper.

2 A relativistic superfluid in a rotating cylinder

1. Let a superfluid is contained in a cylindrical vessel rotating along the axis x^3 with a constant angular velocity ω which is assumed to be large enough for existence of a vortex array in a superfluid, while the distance between neighboring vortices is small in comparison with the size of container R_0 . The equilibrium condition is determined by a free energy minimization[1]:

$$\delta F = \delta E - \omega^i \delta L_i = 0 \quad (1)$$

where

$$E = \int T_0^0 d^3R \quad \omega^i = (0, 0, \omega) \quad (2)$$

is the energy, while

$$L_i = \varepsilon_{ijk} L^{jk} \quad L_3 = -L^{12} \quad (3)$$

and [4]

$$L^{jk} = \int (x^j T^{k0} - x^k T^{j0}) d\phi R dR \quad (4)$$

is the angular momentum and its tensor form, respectively. The superfluid velocity $V(R)$ at distance R from the axis of rotation follows immediately from the constraint $\delta F = 0$.

2. In general the liquid and the vortices are not separated [3] due to the Lagrangian dependence Λ on the amplitude

$$h^2 = h^\nu h_\nu \quad (5)$$

of the helicity vector

$$h^\nu = \frac{1}{2} \varepsilon^{\nu\alpha\beta\gamma} \mu_\alpha W_{\beta\gamma} \quad (6)$$

which, as we see, is the cross product of chemical potential 1-form $\mu_\alpha = \mu u_\alpha$ and vorticity 2-form $W_{\beta\gamma}$. If $\partial\Lambda/\partial h = 0$, the liquid and the vortices can be considered separately (as in the frames of the dilatonic model [3], which may be treated as an analogue of the two-constituent superfluid model [5]). Namely, the particle number current n^ν is collinear to the chemical potential flow: $n^\nu = nu^\nu = \Phi^2 \mu^\nu$. This situation occurs in the weak vorticity limit which takes place inside a typical neutron star [2]. Thus the energy-momentum tensor of superfluid will be [3]

$$T_\rho^\nu = (\rho + P) u^\nu u_\rho - \frac{\lambda}{W} W^{\nu\sigma} W_{\sigma\rho} - \Psi g_\rho^\nu \quad (7)$$

where

$$\lambda = K \Phi^2 \quad \Phi^2 = \frac{n}{\mu} \quad (8)$$

while pressure P of the superfluid constituent and its rest-mass density ρ satisfy the relation $\mu n = \rho + P$ pertaining to a perfect fluid. The second term $\frac{\lambda}{W}$ in the right side of (7) is negligible in the weak vorticity limit, while the pressure function $\Psi \rightarrow P$.

3. For a vortex line situated at the distance R from the axis of the vessel it is convenient to switch to the reference frame rotating with the superfluid medium at the velocity $V(R)$. The metric in the rotating frame has the form [6, 7]

$$ds^2 = - (1 - \omega^2 R^2) dt^2 - 2\omega R d\varphi dt + dR^2 + R^2 d\varphi^2 + dz^2 \quad (9)$$

On account of small b (with respect to the size of the vessel) the metric in the vicinity $O(R)$ of the vortex line can be regarded as flat. Thereby, we can also introduce the local cylindrical coordinates (t, r, ϕ, z) and calculate the invariant circulation integral [3]

$$\frac{1}{2\pi} \int_{\partial U} \mu_\nu dx^\nu = \kappa = \hbar n \quad (10)$$

which determines the local superfluid velocity $v_\phi(r)$ round the vortex

$$v_\phi(r) = \frac{\mu_\phi}{m} = \frac{\kappa}{mr} = \frac{nl_c}{r} \quad (11)$$

where m is the mass of the boson pair and $l_c = \hbar/(mc)$ is the Compton length. Note that the superfluid velocity (defined as a gradient of the bose-condensate wave function phase [8]) does not coincide with the usual four-velocity

$$u_\nu = \frac{\mu_\nu}{\mu} = \frac{m}{\mu} v_\nu \quad (12)$$

This solution (locally) satisfies the irrotationality condition [5, 8]

$$\text{rot } \vec{v}_s = 0 \quad \mu_0 = \text{const} \quad (13)$$

(where

$$\text{rot } \vec{v} \equiv \frac{1}{R} \partial_R (vR) \quad (14)$$

in cylindrical coordinates) which takes no place in the global sense, particularly on the axis of the vortex line.

Although (in the local reference frame pinned to the vortex) the averaging over domain $O(R)$ yields zero momentum

$$\langle \mu_\phi \rangle = 0 \quad (15)$$

the presence of a vortex results in the global rotation of the superfluid: $V(R) \neq 0$. The world sheet Σ of the vortex line plays the role of support of the vorticity 2-form $W_{\nu\varrho}$ ($W_{\nu\varrho} \neq 0$ on Σ). Taking into account the link (11) between the superfluid velocity v_ϕ and the flow chemical potential μ_ϕ and calculating the integral (10) in the laboratory reference frame, we have

$$\kappa = \frac{1}{2\pi} \oint_{\partial U} \mu_\nu dx^\nu = \frac{1}{2\pi} \int_U W_{\nu\varrho} d\sigma^{\nu\varrho} = \frac{m}{2\pi} |\text{rot } \vec{v}| \pi b^2 \quad (16)$$

where $|\text{rot } \vec{v}|(R)$ is the value averaged over domain $U(R)$. Indeed, the global velocity field $\vec{V}(R)$ does not satisfy the irrotationality condition. Formula (16) implies relation between the velocity field and the distance between the vortices:

$$b^2 = \frac{2\kappa}{m|\text{rot } \vec{v}|} \quad (17)$$

Equations (10)-(13) also imply

$$\mu^2 = -\mu^e \mu_e = \mu_*^2 - \frac{\kappa^2}{r^2} \quad (18)$$

where $\mu_*(R) = \sqrt{\mu^0 \mu_0} = \mu_0 / \sqrt{g_{00}(R)}$ and μ is invariant and can be treated as the rest-frame chemical potential. The four-velocity (12) constructed from (18) will be

$$u_\nu = \frac{1}{\sqrt{1-w^2}} (-1, 0, w, 0) \quad w(r) = \frac{\kappa}{\mu r} = \frac{r_c}{r} \quad (19)$$

where the modified Compton length is

$$r_c = \frac{\kappa}{\mu} = l_c \frac{m}{\mu} \quad (20)$$

4. Followed the non-relativistic procedure [1], we can take into account the possible non-uniform distribution of vortices if: calculate the energy of a single vortex line and multiply it on the density of the vortices

$$N(R) = \frac{1}{\pi b^2} = \frac{m |\text{rot} \vec{v}|}{2\pi \kappa} \quad (21)$$

Substituting (19) in (7) we find the energy

$$E(R) = \int T_0^0 r dr = \int_a^b \left(\frac{\rho(r)}{1-w^2} + \frac{w^2 P}{1-w^2} \right) r dr \quad (22)$$

of a single vortex calculated in the local reference frame (associated with point R). Of course, due to the local velocity $w(r) \neq 0$ round the vortex (19) the mass density $\rho(r)$ deviates from the proper *rest* mass density ρ_s . Taking into account the evident relation $\rho(r) = \rho_s \sqrt{1-w^2}$ between them and Eq. (22), we obtain

$$E(R) = \pi \left[\rho_s \left(b \sqrt{b^2 - r_c^2} - a \sqrt{a^2 - r_c^2} \right) + \rho_s r_c^2 \ln \left(\frac{b + \sqrt{b^2 - r_c^2}}{a + \sqrt{a^2 - r_c^2}} \right) + P r_c^2 \ln \left(\frac{b^2 - r_c^2}{a^2 - r_c^2} \right) \right] \quad (23)$$

where a is the inner cutoff radius determined from the kinetics concepts. The typical values for neutron stars are [2]: $b \sim 10^{-4}$ cm, while a is, at least,

several times greater than $r_c \sim 1$ fm (we do not consider the exotic cluster vortices [9] with $n \sim 10^{12}$). Thus, Eq. (23) is simplified so that

$$E(R) \cong \pi \left[\rho_s b^2 + \rho_s r_c^2 \ln \left(\frac{2b}{a + \sqrt{a^2 - r_c^2}} \right) + P r_c^2 \ln \left(\frac{b^2}{a^2 - r_c^2} \right) \right] \quad (24)$$

Multiplying (24) by (21), we find the superfluid energy density

$$\varepsilon(R) = \rho_s + \rho_s \frac{r_c^2}{b^2} \ln \left(\frac{2b}{a + \sqrt{a^2 - r_c^2}} \right) + P \frac{r_c^2}{b^2} \ln \left(\frac{b^2}{a^2 - r_c^2} \right) \quad (25)$$

at point R , or (more roughly)

$$\varepsilon(R) = \rho_s + (\rho_s + 2P) \frac{r_c^2}{b^2} \ln \left(\frac{b}{a} \right) \quad (26)$$

It is the mean value (we may define it also as $\varepsilon_s \equiv \langle T_0^0 \rangle$) of the energy density in the vicinity $O(R)$, for (25) bears no dependence on the local coordinate r and, hence, does not reflect the fine structure of the vortex cell. The energy ε measured in the laboratory reference frame will be given merely by formula [6, 7]

$$\varepsilon = \frac{\varepsilon(R)}{\sqrt{1 - V_R^2}} \quad (27)$$

where Eq. (15) was taken into account and the global ordinary velocity V_R is resulted from the global superfluid velocity (do not mix it with the local filed v_ϕ)

$$v = \frac{V_R}{\sqrt{1 - V_R^2}} \quad (28)$$

which coincides with V_R in the non-relativistic limit.

The mass density ρ_L in the laboratory frame is expressed through mass density in the rotating frame as $\rho_R = \rho_L \sqrt{1 - V^2}$, while ρ_R is determined as the averaging over domain $O(R)$:

$$\rho_R = \langle \rho(r) \rangle = \int_a^b \frac{\rho_s}{\sqrt{1 - V_R^2}} r dr \simeq \rho_s \left(1 + \frac{r_c^2}{b^2} \ln \frac{b}{a} \right) \quad (29)$$

Indeed, this term, which has appeared in (26), includes contribution from the vortex.

Thereby

$$\varepsilon_R = \frac{\rho_s}{\sqrt{1-V_R^2}} + \frac{\rho_s + 2P}{\sqrt{1-V_R^2}} \frac{r_c^2}{b(R)^2} \ln \frac{b(R)}{a} \quad (30)$$

while, according to (11), (17), and (20),

$$\frac{r_c^2}{b^2} = \frac{\kappa}{2} |\text{rot} \vec{v}| \frac{m}{\mu^2} = \frac{r_c}{2} |\text{rot} \vec{v}| \frac{m}{\mu} = \frac{\chi}{4\pi} |\text{rot} \vec{v}| \quad (31)$$

where

$$\chi = 2\pi \hbar n \frac{m^2}{\mu^2} \quad (32)$$

reduces to the usual circulation quantum [1] in the non-relativistic limit. We shall bellow a more convenient expression

$$\alpha = \frac{\chi}{8\pi} = cr_c \frac{m}{4\mu} = cl_c \frac{n}{4} \frac{m^2}{\mu^2} \quad (33)$$

instead.

5. The total angular momentum density [4]

$$(0, 0, L_3) \rightarrow L^{12} = M^{12} + S^{12} \quad (34)$$

includes spin contribution of the vortices S^{jk} and the orbital part M^{jk} from the rotating fluid itself. The latter may be calculated (in the laboratory frame) by formula (4):

$$M^{12} = \int x^1 T^{20} dV = \int (\rho_R + P) \frac{V_R R}{\sqrt{1-V_R^2}} d^3 R \quad (35)$$

while the spin contribution from a single vortex is determined by integral (10):

$$s^{12} = \frac{1}{2\pi} \oint_{\partial O} \mu_\nu dx^\nu = \kappa \quad (36)$$

It is invariant with respect to transformation from the rotating to the laboratory reference frame. This is derived immediately from the constraints [6, 7]

$$s^\mu s_\mu = -s^2 \quad p^\mu s_\mu = 0 \quad (37)$$

which are the same in any reference frame; here p^μ is the momentum. Since in the rotating frame

$$s_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho} s^{\nu\rho} = (s^0, 0, 0, s^z) \quad p^\mu = (p^0, 0, p^\phi, 0) \quad (38)$$

and $p^0 \neq 0$, we find, by means of (37) and metric (9), that $s^0 = 0$. Hence, $s^2 = (s^z)^2$ and, in the laboratory frame

$$\check{s}_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho} \check{s}^{\nu\rho} = (\check{s}^0, 0, 0, \check{s}^z) \quad p^\mu = (\check{p}^0, 0, \check{p}^\phi, 0) \quad (39)$$

The identity (37) then implies that

$$\check{p}^0 \check{s}_0 + \check{p}^\phi \check{s}_\phi = \check{p}^0 \check{s}_0 = 0 \quad (40)$$

because $\check{s}_0 = 0$, according to (40), and the metric is defined as (9). Therefore, also $\check{s}_0 = 0$ and, hence, $\check{s}_z^2 = s^2 = s_z^2$.

Thus, we can define the spin density

$$\frac{\rho_R}{m} s^{12} \quad (41)$$

measured in the rotating reference frame (related to the point R).

Hence,

$$S^{12} = \int \frac{\kappa \rho_R / m}{\sqrt{1 - V_R^2}} d^3 R \quad (42)$$

will be the total spin in the laboratory frame [6, 7]. Combining Eqs. (34), (35), (42) and (29), (31), we get the total angular momentum

$$\frac{L}{2\pi\rho_s} = \int \frac{RdR}{\sqrt{1 - V_R^2}} \left\{ \left[\left(1 + \frac{r_c^2}{b^2} \ln \frac{b}{a} \right) + \frac{P}{\rho_s} \right] V_R R - \left(1 + \frac{r_c^2}{b^2} \ln \frac{b}{a} \right) \frac{\kappa}{m} \right\} \quad (43)$$

6. The total free energy is constructed from (1), (30), (35), (34), (39) so

$$\frac{F}{2\pi\rho_s} = \int \left\{ \left(1 + \frac{\rho_s + 2P}{\rho_s} \frac{r_c^2}{b^2} \ln \frac{b}{a} \right) - \left[\left(1 + \frac{r_c^2}{b^2} \ln \frac{b}{a} \right) + \frac{P}{\rho_s} \right] \frac{v_R \omega R}{c} - \left(1 + \frac{r_c^2}{b^2} \ln \frac{b}{a} \right) \frac{\kappa \omega}{m} \right\} \frac{RdR}{\sqrt{1 - V_R^2}} \quad (44)$$

that is

$$\frac{F}{2\pi\rho_s} = \int R dR \left\{ J\gamma + (v\omega R - \Xi\gamma) \frac{\chi}{8\pi} |\text{rot}\vec{v}| \ln \frac{\pi |\text{rot}\vec{v}| a^2}{\chi} - \Gamma v\omega R \right\} \quad (45)$$

where, in the lighth of (31),

$$\gamma = \frac{1}{\sqrt{1 - V_R^2}} = \sqrt{1 + v^2} \quad (46)$$

the parameters

$$J = 1 - \frac{\kappa\omega}{m} \quad \Xi = 1 + \frac{2P}{\rho_s} - \frac{\kappa\omega}{m} \quad (47)$$

and the equation of state (EOS) index

$$\Gamma = 1 + \frac{P}{\rho_s} \quad (48)$$

tend to a unit in the non-relativistic limit. Note that

$$\xi = \frac{\kappa\omega}{m} = \frac{\hbar\omega n}{mc^2} = \frac{l_c}{R_2} \frac{n\omega}{\omega_M} \quad \omega < \omega_M = \frac{c}{R_2} \quad (49)$$

implies the ratio of the spin constituent of the rotation energy per particle (one may call it as "rotation quantum") to its rest-mass energy. It does not exceed the ratio of the Compton length to the radius of the star and occurs to be extremely small, namely, it is evaluated as 10^{-21} even at angular velocity $\omega = 10^3$ which is rather high for neutron stars. One may only mediate at application to a hydrodynamic model of rotating nuclei emphasized e.g. in [11].

3 The equation of motion

Varying expression (45) over δv , as shown in Appendix in details, we obtain the equation of motion

$$J \frac{v}{\sqrt{1 + v^2}} - \omega R \Gamma + \alpha \left[\Xi \sqrt{1 + v^2} - v\omega R \right] \frac{\partial_R |\text{rot}\vec{v}|}{|\text{rot}\vec{v}|} + \alpha \Xi \frac{v}{\sqrt{1 + v^2}} \left[|\text{rot}\vec{v}| - \frac{v}{R} \ln \frac{ea^2}{b^2} \right] - \alpha \omega R |\text{rot}\vec{v}| = 0 \quad (50)$$

Or, including the speed of light explicitly

$$\omega R \Gamma - J \frac{v}{\sqrt{1+v^2/c^2}} = \alpha \left[\Xi \sqrt{1+v^2/c^2} - \frac{1}{c^2} v \omega R \right] \frac{\partial_R |\text{rot} \vec{v}|}{|\text{rot} \vec{v}|} + \alpha \frac{1}{c^2} \Xi \frac{v}{\sqrt{1+v^2/c^2}} \left(\partial_R v - \frac{v}{R} \ln \frac{b^2}{a^2} \right) - \alpha \quad (51)$$

While its dimensionless form

$$\left(J \frac{v}{\sqrt{1+v^2}} - x \Gamma \right) W = \xi_* \left\{ \Xi \frac{v}{\sqrt{1+v^2}} \left(\frac{v}{x} \ln \frac{a^2}{b^2} - W \right) W + x W^2 + \left(W - \Xi \sqrt{1+v^2} \right) \partial_x W \right\} \quad (52)$$

with

$$\xi_* = \xi \frac{m^2}{4\mu^2} \quad W = \frac{1}{x} \partial_x (xv) \sim |\text{rot} \vec{v}| \quad x = \frac{\omega R}{c} \quad (53)$$

is also convenient for further discussion.

When simultaneously $c \rightarrow 0$ and $\Gamma \rightarrow 1$, Eq. (51) reduces to the well-known non-relativistic equation [1]

$$v - \omega R + \alpha \frac{\partial_R |\text{rot} \vec{v}|}{|\text{rot} \vec{v}|} = 0 \quad (54)$$

for the rotating superfluid helium.

While the non-relativistic equation (54) was describing the solid-body rotation $v = \omega R$ in the inner domain ($R < R_i$) and the irrotational motion

$$\text{rot} \vec{v} = 0 \quad (55)$$

in the outer domain ($R_i < R < R_2$), the general equation (50)-(52) also determines an irrotational solution (55) but it does admit any solid-body rotation in the strict sense. However, when the right side of (52) is small, the left side is splitted into a product of these two solutions which are connected in the transition region ($R \simeq R_i$) whose width l is small with respect to R_i and $R_2 - R_i$.

The intermediate region (between the solid-body rotation at small R and the irrotational motion in the outer layers) may become relatively broad at significant right side of (52). On account of small ξ_* , this may occur at ultra-relativistic velocities $v \sim 1/\xi \gg 1$. Even if we apply the present analysis to the cluster vortices with $n \sim 10^{12}$, the lowest value will be

$$v > 10^7 \quad (56)$$

Although one could consider this situation in the frames of the hydrodynamic nuclear model, it does not appear when we study the real neutron stars. Therefore, except for the main transition region, the solution is determined by equation

$$\left\{ \frac{v}{\sqrt{1+v^2}} - \omega R \Gamma \right\} \text{rot} \vec{v} = O(\xi) \quad (57)$$

which yields

$$\frac{v_+}{\sqrt{1+v_+^2}} = V_+ = \Gamma \omega R \quad \sqrt{1+v_+^2} = \frac{1}{\sqrt{1-(\Gamma \omega R)^2}} \quad (58)$$

or, the angular velocity

$$\Omega = \Gamma \omega \quad (59)$$

of the solid-body rotation in the inner region. It differs from the non-relativistic rotation by multiple $\Gamma = 1 + \frac{P}{\rho_s}$. So, for an ultrarelativistic equation of state ($\rho = 3P$) a superfluid rotates at angular velocity $\Omega = \frac{4}{3}\omega$ rather than $\Omega = \omega$. For a stiff EOS ($\rho = P$) the superfluid angular velocity exceeds twice the angular velocity of the vessel. The conclusion applied to the neutron stars is evident: the angular velocity of the superfluid core Ω differs sufficiently from the angular velocity of the crust ω .

As for the irrotational solution

$$v_- = \frac{Q}{R} \quad (60)$$

of (57) in the outer region, it must obey the boundary condition $v(R_2) = \omega R_2 / \sqrt{1 - \omega^2 R_2^2 / c^2} = Q / R_2$ and, hence, be defined ultimately

$$v_- = \frac{\omega R_2}{\sqrt{1 - \omega^2 R_2^2 / c^2}} \frac{R_2}{R} \quad (61)$$

That is

$$v_- = \frac{x_0^2}{x \sqrt{1 - x_0^2}} = \frac{q}{x} \quad (62)$$

in the dimensionless form, where

$$q \equiv \frac{x_0^2}{\sqrt{1 - x_0^2}} \quad x_0 \equiv \frac{\omega R_2}{c} \quad (63)$$

Indeed, in the light of formula (58) only $x_0 \leq 1/\Gamma$ has physical sense.

4 The boundary between two solutions

4.1 The extremum

In order to find the boundary R_i between the two types of motion, determined by formulae (61) and (58), respectively, we verify the free energy (64) over unknown R_i which must minimize F . The latter, on account of the thin intermediate region (whose size is determined by usual expression [1] multiplied by Γ), can be splitted into a sum

$$F = \int_{R_i}^{R_2} f_-(R) R dR + \int_{R_1}^{R_i} f_+(R) R dR \quad (64)$$

of irrotational $f_-(R) = f[v_-(R)]$ and solid-body contribution $f_+(R) = f[v_+(R)]$, which we present in convenient dimensionless form

$$\frac{F_-}{2\pi\rho_s} = \frac{c^2}{\omega^2} \int_{x_i}^q x dx \left(\sqrt{1 + \frac{q^2}{x^2}} - \Gamma q \right) \quad (65)$$

$$\frac{F_+}{2\pi\rho_s} = \frac{c^2}{\omega^2} \int_0^{x_i} x dx \left\{ \frac{1}{\sqrt{1 - \Gamma^2 x^2}} - \frac{\Gamma^2 x^2}{\sqrt{1 - \Gamma^2 x^2}} + o(\alpha) \right\} \quad (66)$$

where quantum contribution $o(\alpha)$ is, evidently, proportional to the quantum number n (the non-linear dependence appears at ultrarelativistic rotation, when the quantum term is sufficient, that may occur in nuclear hydrodynamics).

The condition of extremum

$$\frac{dF}{dR} \Big|_{R=R_i} = -f_+(R_i) R_i + f_-(R_i) R_i = 0 \quad (67)$$

or

$$f_+(x_i) = f_-(x_i) \quad (68)$$

allows to determine radius $R_i = x_i c / \omega$ without direct calculation of the total free energy (64), (1).

4.2 The non-relativistic velocities

In principle, we may confine ourselves with the non-relativistic velocities of rotation (while the ratio P/ρ is sufficient), because $\frac{\Omega R_2}{c} \leq \frac{1}{30}$ at typical values $\Omega \leq 10^3 \text{ s}^{-1}$ and $R_2 \simeq 10 \text{ km}$ for the majority of neutron stars. Even for very high Ω , approaching to 10^4 s^{-1} [10], the quantity $x^2 = \frac{1}{9}$ is small.

At non-relativistic velocities the free energy (64) reduces to

$$\frac{F_{nr}}{2\pi\rho_s} = \int R dR \left[\frac{v^2}{2} - \Gamma v \omega R + \Xi \alpha |\text{rot} \vec{v}| \ln \frac{b^2}{a^2} \right] \quad (69)$$

and its irrotational and solid-body contribution are specified immediately:

$$f_-(R) = f[v_-(R)] = \frac{\omega^2 R_2^2 R_2^2}{2R} - \Gamma \omega^2 R_2^2 R \quad (70)$$

$$f_+(R) = f[v_+(R)] = \frac{\Omega^2 R^3}{2} + \Xi \alpha R |\text{rot} \vec{v}| \ln \frac{b^2}{a^2} - \Gamma \Omega \omega R^3 \quad (71)$$

Substituting them in extremum condition (67) leads to

$$\omega^2 \frac{R_2^2}{R_i^2} - 2\Gamma \omega^2 = -\frac{\Omega^2 R_i^2}{R_2^2} + \frac{2\Xi \alpha}{R_2^2} \cdot 2\Omega \ln \frac{b^2}{a^2} \quad (72)$$

and, after plain arithmetics, yields

$$R_i = \frac{R_2}{\sqrt{\Gamma}} - \sqrt{\frac{2\Gamma - 1}{\Gamma} \frac{\alpha}{\omega} \ln \frac{b^2}{a^2}} \quad (73)$$

The quantum term in the right side of (73) is at least several orders less than R_2 . But for superfluid helium [1] ($\Gamma - 1 = \frac{P}{\rho} \sim 10^{-16}$) it defines a narrow band

$$R_2 - R_i = \sqrt{\frac{\alpha}{\omega} \ln \frac{b^2}{a^2}} \sim 10^{-2} \text{ cm} \quad (74)$$

of irrotational motion near the walls of the container. For a relativistic matter, whose pressure P is compared with its energy density ρ , the deviation of the radius

$$R_i \cong \frac{R_2}{\sqrt{\Gamma}} \quad (75)$$

from R_2 is significant. For an ultrarelativistic matter $R_i = 0.87R_2$, and $R_i = 0.71R_2$ for a stiff matter ($\Gamma = 2$). Therefore, a $35 \div 65 \%$ portion of the total volume (which is proportional to R^2 in cylindrical symmetry) is free of vortices.

4.3 The relativistic velocities

Substituting (65) and (66) in (68) we obtain the equation

$$\sqrt{1 + \frac{q^2}{x_i^2}} - \Gamma q = \sqrt{1 - \Gamma^2 x_i^2} + o(\alpha) \quad (76)$$

for dimensionless $x_i = \omega R_i / c$. As we have mentioned above, the quantum term $o(\alpha)$ does not play significant role in defining R_i at the relativistic equation of state. Solving (66) without $o(\alpha)$, we find that

$$\Gamma x_i^2 = q \left(\sqrt{1 + \frac{\Gamma^2 q^2}{4}} - \frac{\Gamma q}{2} \right) = \frac{x_0^2}{2(1 - x_0^2)} \left(\sqrt{4 - 4x_0^2 + \Gamma^2 x_0^4} - \Gamma x_0^2 \right) \quad (77)$$

generalizes (75) at relativistic velocities. The ratio x_i/x_0 always decreases with the growth of x_0 implying that relativistic rotation tends to diminish R_i . And the ratio

$$\frac{x_i^2}{x_0^2} = \frac{1}{\Gamma^2 - 1} \left(\sqrt{\Gamma^2 - \frac{3}{4}} - \frac{1}{2} \right) \quad (78)$$

corresponds to the $\Gamma x_0 \rightarrow 1$. Note, however, that $x_i \rightarrow x_0$ as soon as $\Gamma \rightarrow 1$; meanwhile, the smallest ratio $x_i/x_0 = 0.66$ is achieved at $\Gamma = 2$.

5 The angular momentum

5.1 The general expression

The total angular momentum (43) can be presented as a sum of classical and quantum terms

$$\frac{L}{2\pi\rho_s} = \int R dR \Gamma v R + L(\alpha) \quad (79)$$

where the quantum contribution $L(\alpha)$ is negligible in comparison with the first term, because the right side of equation (52) is small (the quantum term is sufficient only at ultrarelativistic rotation). Therefore, omitting it and substituting the velocity fields (61) and (58) in (79), we get the dimensionless

expression

$$\frac{L}{2\pi\rho_s}\frac{\omega^2}{\Gamma c^2} \simeq \int_0^{x_i} x^2 dx \frac{\Gamma x}{\sqrt{1-\Gamma^2 x^2}} + \int_{x_i}^{x_0} x^2 dx \frac{q}{x} \quad (80)$$

easily integrated up to

$$\frac{L}{2\pi\rho_s}\frac{\omega^2}{\Gamma c^2} = \frac{2 - \sqrt{1 - \Gamma^2 x_i^2} (2 + \Gamma^2 x_i^2)}{3\Gamma^3} + \frac{1}{2}q (x_0^2 - x_i^2) \quad (81)$$

where $x_0 = \omega R_2/c$ and x_i are determined by (77).

5.2 The non-relativistic velocities

The non-relativistic angular momentum can be derived either from formula (81) or from (79). Substituting there (61) and (58), we have

$$\frac{L_s}{2\pi\rho} = \int_0^{R_i} \Omega R \Gamma \omega R R dR + \int_{R_i}^{R_2} \frac{\omega R_2^2}{R} \Gamma \omega R R dR \quad (82)$$

that is

$$\frac{L_s}{2\pi\rho} = \frac{1}{2}\omega^2 R_2^4 \left(\Gamma - \frac{1}{2} \right) \quad (83)$$

and always exceeds the non-relativistic value $\frac{L}{2\pi\rho} = \frac{1}{4}\omega^2 R_2^4$. Their ratio is 5/3 for $\Gamma = 4/3$ and 3 for $\Gamma = 2$. Formula (83) may be also compared with the angular momentum

$$\frac{L_n}{2\pi\rho} = \frac{1}{4}\omega^2 R_2^4 \Gamma^2 \quad (84)$$

of normal fluid rotating as a solid body with the same radius R_2 and angular velocity ω . If normal matter rotating with angular velocity ω_n transfers to superfluid with the same angular momentum, its angular velocity

$$\frac{\omega_s^2}{\omega_n^2} = \frac{\Gamma^2}{2\Gamma - 1} \quad (85)$$

will be always greater than ω_n . The first relativistic correction

$$\frac{\omega_s^2}{\omega_n^2} \cong \frac{\Gamma^2}{2\Gamma - 1} + \frac{\Gamma^3}{2\Gamma - 1} \left(\Gamma - \frac{4\Gamma - 3}{(2\Gamma - 1)^{3/2}} \right) \frac{x_0^2}{3} + O(x_0^4) \quad (86)$$

does not contradict to (85) states the same, namely that the superfluid rotates also *faster* than the normal fluid with the same angular momentum. Meanwhile,

$$\frac{\omega_s^2}{\omega_n^2} \rightarrow 1 \quad (87)$$

when $\Gamma \rightarrow 1$. Indeed, the difference between ω_s and ω_n is not negligible at relativistic EOS, because $\Gamma - 1$ is sufficient. This forms the background for further application to neutron stars and pulsar glitches.

5.3 The relativistic velocities

The angular momentum of relativistic superfluid (81) differs from its expression

$$\frac{L_n}{2\pi\rho_s} \frac{\omega^2}{\Gamma c^2} = \frac{2 - \sqrt{1 - \Gamma^2 x_0^2} (2 + \Gamma^2 x_0^2)}{3\Gamma^3} \quad x_0 \leq \frac{1}{\Gamma} \quad (88)$$

for normal matter with the same parameters, i.e. density ρ_s , equation of state (indicates Γ), external radius (dimensionless x_0). The angular momentum (88), as well as (81), increases with growth of x_0^2 . While always $L_s \leq L_n$, especially at relativistic rotation. Indeed, they coincide at $\Gamma = 1$ (for we have omitted the negligible quantum contribution), while their ratio (L_s/L_n) at different Γ is given in the table below.

$\frac{\Gamma x_0}{L_s/L_n}$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$\Gamma = 1.1$	0.992	0.99	0.989	0.987	0.984	0.979	0.973	0.964	0.947	0.911	0.694
$\Gamma = 1.2$	0.972	0.968	0.964	0.958	0.95	0.94	0.926	0.907	0.876	0.82	0.60
$\Gamma = 1.6$	0.86	0.85	0.83	0.82	0.80	0.78	0.75	0.71	0.67	0.60	0.42
$\Gamma = 2$	0.75	0.735	0.72	0.70	0.68	0.65	0.62	0.59	0.54	0.48	0.33

(89)

Instead of x_0 , the universal variable Γx_0 stands here in the first row.

6 Conclusion

Summarizing, we list the main results. Having derived the equation of motion (50)-(52) for the rotating relativistic superfluid, we find that it may be reduced to a product of two independent solutions (57), namely the solid body

rotation (58) in the inner domain $R < R_i$ and the irrotational motion (61) in the outer domain $R > R_i$ with a relatively thin region where they interfere at $R = R_i$. The splitting (57) into these independent solutions was possible due to a small rotation quantum (49) implying that the velocity of rotation is not extremely high (56), that takes place for all real macroscopic objects, particularly, the neutron stars. Although, considering the nuclei within the hydrodynamic approach, one encounters the opposite situation and has to solve equation of motion (52) numerically.

As for solutions (58) and (61), there should be noted the fact of dependence on the equation of state, besides the relativistic expression for velocity. The *relativistic equation of state* and *relativistic rotation* are characterized by dimensionless quantities $\Gamma = 1 + P/\rho_s$ and $x_0 = \omega R_2/c$, respectively (which, in the non-relativistic limit, become $\Gamma \rightarrow 1$ and $x_0 \rightarrow 0$). However, without regard of rotation rate the relativistic EOS results to difference between the angular velocity (59) of the superfluid – obeying the solid-body rotation (58) within the vessel, – and the angular velocity ω of the vessel itself. In other words, the superfluid rotating as a solid body rotates at angular velocity which Γ times higher than that of the vessel.

The same quantity Γ appears in formula (73), (76) for radius R_i which corresponds to the boundary between the solid-body and irrotational motion. While the non-relativistic value of R_i deviates slightly from R_2 due to the presence of quantum term (74), this quantum term plays an insufficient role when the difference (75), (77) between R_i and R_2 becomes considerable on account of the relativistic equation of state or relativistic rotation. Particularly, the boundary radius R_i is determined merely by Γ for a relativistic matter rotating at low velocity (75). Or, briefly, Eq. (77) states that both relativistic rotation and EOS *shorten* the distance R_i , i.e. the region of irrotation motion is much *wider* than that in non-relativistic helium.

So, the quantum term involved in the relevant non-relativistic formulae becomes insufficient for a relativistic superfluid. Thus, the total angular momentum (79) is given approximately by formula (81). The angular momentum of a slowly rotating relativistic superfluid (83) is always less than the angular momentum (84) of the normal fluid rotating with the same angular frequency. Hence, if the normal matter, rotating with frequency ω_n , transfers into a superfluid state (with angular momentum conserved), its angular velocity ω_s will be increased (85). The general dependence (81) of ω_s/ω_n on the initial angular velocity $x_0 = \omega R_2/c$ and the EOS index Γ is more

complicated as illustrated in table (89). However, the angular momentum of superfluid (81) occurs to be always smaller than that of normal matter (88) and the relativistic rotation contributes to this tendency.

Thus, the equation of state and velocity ωR_2 of a rotating superfluid are of particular importance when they belong to a relativistic range. A plenty of applications (the pulsar glitches, for instance) are expected to be derived from here. And further development in the light of nuclear hydrodynamics or solution with a solid core (that is $R_2 > R > R_1 > 0$) may be also proposed.

7 Appendix: variation of the free energy

Since the rotation quantum (49) is small for usual macroscopic objects, we can rewrite (45) as

$$\frac{F}{\rho_s} = 2\pi \int f R dR \quad f = \gamma J + \alpha |\text{rot} \vec{v}| \ln \frac{a^2}{b^2} \{v\omega R - \psi\gamma\} - \Gamma v\omega R \quad (90)$$

Firstly we specify variation

$$\delta G[v] = \frac{\partial G}{\partial v} \delta v \equiv G' \delta v \quad (91)$$

of an arbitrary function $G(v)$. Hence, the simplest expressions for variations

$$\delta B = \delta(\gamma J - \Gamma v\omega R) = \left(\frac{Jv}{\sqrt{1+v^2}} - \Gamma\omega R \right) \delta v \quad (92)$$

for

$$\gamma = \frac{1}{\sqrt{1-V_R^2}} = \sqrt{1+v^2} \quad (93)$$

The variation of

$$A = -\alpha |\text{rot} \vec{v}| G \ln \frac{\chi}{\pi |\text{rot} \vec{v}| a^2} = \alpha \frac{\partial_R(vR)}{R} G \ln \frac{a^2}{b^2} \quad (94)$$

is performed so:

$$\delta A = \alpha \int \left[\partial_R(\delta v R) G \ln \frac{a^2}{b^2} + \partial_R(vR) \frac{\delta(1/b^2)}{1/b^2} G + \partial_R(vR) \ln \frac{a^2}{b^2} \delta G \right] dR \equiv \delta A_1 + \delta A_2 + \delta A_3 \quad (95)$$

where

$$\ln \frac{a^2}{b^2} = \frac{\partial_R (1/b^2)}{1/b^2} = \frac{\partial_R |\text{rot} \vec{v}|}{|\text{rot} \vec{v}|} \quad \frac{\delta (1/b^2)}{1/b^2} = \frac{\delta |\text{rot} \vec{v}|}{|\text{rot} \vec{v}|} = \frac{\partial_R (\delta v R)}{\partial_R (v R)} \quad (96)$$

Hence,

$$\delta A_2 = \int \partial_R (\delta v R) G dR \quad \delta A_3 = \int \partial_R (v R) \ln \frac{a^2}{b^2} G' \delta v dR \quad (97)$$

The first two terms $\delta A_1 + \delta A_2$ of (95) are simplified as

$$\delta A_1 + \delta A_2 = \int \partial_R (\delta v R) \ln \frac{ea^2}{b^2} G dR = \delta v R \ln \frac{ea^2}{b^2} G|_{R_1}^{R_2} - \int \delta v R \partial_R \left(\ln \frac{ea^2}{b^2} G \right) dR \quad (98)$$

The total variation of (95) will be

$$\delta A = \delta v R \ln \frac{ea^2}{b^2} G|_{R_1}^{R_2} + \int \delta v \left\{ \partial_R (v R) \ln \frac{a^2}{b^2} G' [v] - R \partial_R \left(G [v] \ln \frac{ea^2}{b^2} \right) \right\} dR \quad (99)$$

Taking into account that variation δv vanishes at the edges of the vessel (i.e. at $R = R_1$ and $R = R_2$), and adding (92) to the latter formula, we find the total variation of free energy

$$\delta F = \int \delta v \left[\frac{J v}{\sqrt{1 + v^2}} - \Gamma \omega R + G' \left(v \ln \frac{a^2}{b^2} - R \partial_R v \right) - G R \frac{\partial_R |\text{rot} \vec{v}|}{|\text{rot} \vec{v}|} \right] dR \quad (100)$$

which for arbitrary δv and $G[v] = v \omega R - \psi \gamma[v]$ yields equation of motion (50). The relevant non-relativistic version is immediately obtained if we put $G \equiv 1$.

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